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Crossover and the Breakdown of Hyperscaling in Long-Range Bond Percolation

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The problem of long-range bond percolation (LRBP) is studied both with scaling arguments and simulation methods. New scaling relations are proposed for the mean-field region and the fractal properties of the LRBP clusters are also investigated.

KEY WORDS: Hyperscaling; mean-field; percolation; crossover; fractal dimension; Leath algorithm.

1. INTRODUCTION

Mean-field percolation has proven to be somewhat more subtle than originally expected. The introduction by Newman and Schulman[1,2] of the possibility of an infinite number of spanning clusters at or above the percolation threshold and the subsequent development of this idea by Coniglio,[3] Aharony *et al.*,[4] and de Arcangelis[5] for nearest-neighbor bond percolation above six dimensions demonstrates the subtlety of this problem. In addition, the presence of a mean-field scaling regime in long-range bond percolation (LRBP) is generally accepted, whereas the situation in the long-range site problem remains unresolved in the literature.[6–11]

Recently it has become clear that critical droplets in nucleation near the mean-field spinodal can be described as LRBP clusters.[12–15] Monte Carlo studies of Ising models in two dimensions[16,17] have obtained a Hausdorff dimension[18] of d (the Euclidean dimension) for these droplets (or clusters) rather than the value of 4 one would have expected from a

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naive extension of the results obtained for models with short-range bonds in dimensions greater than six. This has motivated the present study of LRBP.

In this work we have employed both simulations and scaling arguments to investigate the mean-field regime of LRBP when $p \lesssim p_c$, as well as some aspects of the crossover to nearest-neighbor bond percolation. Our main results are that certain scaling relations must be modified in the LRBP limit and that the fractal[19] dimension d_f, which we distinguish from the Hausdorff dimension, depends on the way it is measured. In this work we will always refer to the fractal dimension as the power associated with the way the mass of clusters M whose radius of gyration is of the order of the connectedness length ξ scales with ξ; i.e., $M \sim \xi^{d_f}$. The Hausdorff dimension d_H will be used to refer to the way the mass scales with a running length. That is, if the density of the cluster ρ_1 is plotted as a function of the radial coordinate l and $\rho_1 \sim l^{d_H - d}$, then d_H is the Hausdorff dimension.

The remainder of this paper is divided into five sections. In Section 2 we establish the fact that our simulations are well within the mean-field regime. We show that all critical exponents are classical and argue that the LRBP clusters are isomorphic to those in bond percolation on a Bethe lattice. In Section 3 we describe the geometrical mechanism which enables LRBP to cross over to the asymptotic regime and show how it is related to the breakdown of hyperscaling. Section 4 contains our analysis of the fractal properties of clusters. We summarize our results in Section 5 and briefly discuss the relation between our work and that of Coniglio,[3]

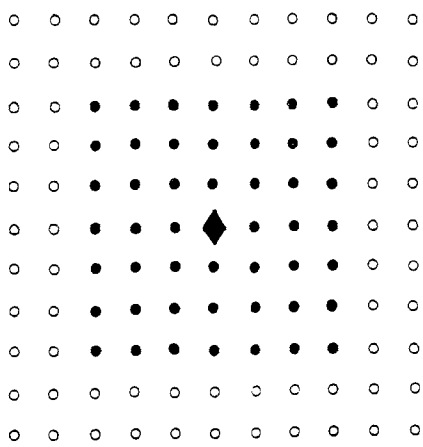


Fig. 1. Neighborhood of a site in two dimensions. Sites which can be directly connected by a bond to the central site are indicated by solid dots. In this example $R = 3$.

Aharony *et al.*,[4] and de Arcangelis[5] concerning the breakdown of hyperscaling for nearest-neighbor bond percolation above six dimensions. Finally, we include a description of our algorithm in the Appendix.

Before proceeding, we define the LRBP problem we use in the simulations. We are dealing with a random bond percolation problem defined on a hypercubic lattice with dimension $d < 6$. A site can be directly connected by a bond to any site within a surrounding neighborhood. The boundary of this neighborhood is a box with sides parallel to the axes of the lattice. Figure 1 shows a neighborhood about an arbitrary site in two dimensions. The length of a side of the box is defined to be $2R+1$ lattice spacings. R is an integer called the "range of the bonds." The number of neighbors for a given value of R is $(2R+1)^d - 1$. All bonds are not the same size; their lengths vary from a minimum of one lattice spacing to a maximum of $R\sqrt{d}$ lattice spacings. In our simulations R ranges from 1000 to 20,000 lattice spacings.

2. THE MEAN-FIELD REGIME IN LRBP

In this section we present evidence that LRBP with a sufficiently large coordination number exhibits mean-field scaling over a certain range of bond probabilities below p_c. We discuss the similarities between the LRBP clusters in this regime and clusters in a different percolation problem: bond percolation on a Bethe lattice. It appears that the intrinsic structure of clusters in both problems is identical, but the way in which clusters are "arranged" on their respective lattices is quite different. We have measured the three critical exponents γ, τ, and ν (defined below). The numbers obtained are the classical values 1, 5/2, and 1/2, respectively. These results are independent of both the range of the bonds R and the dimension of space d as long as we remain in the mean-field regime. Finally, we demonstrate that the parameters we choose are always such that the well-known Ginzburg criterion for classical behavior is satisfied.

In order to measure critical exponents, we must know the value of p_c to within a reasonable degree of accuracy. The approximate values of $\Delta p = (p_c - p)/p_c$ which are in the range of our measurements must not differ appreciably from the true ones. Empirically, we should be able to test the approximation simply by looking for deviation from power-law behavior as Δp decreases. We will adopt this view and choose $1/(z-1)$ as an approximate value for p_c (here z is the coordination number). The motivation for this choice comes in part from the work of Erdös and Rényi,[20] who studied a graph of N sites in which every pair of sites has a probability p of being directly connected by a single bond. When $p \gtrsim 1/N$ there is a nonzero fraction of sites which belong to a single cluster in the

limit $N \to \infty$. The large coordination numbers in the LRBP simulations lead us to expect a similar type of behavior; namely, that near $p = 1/(z-1)$ any given site should have a nonzero probability of being connected to a non-zero fraction of its neighbors.

Another reason to take $1/(z-1)$ to be the approximate value of p_c is that it is the true value of p_c for bond percolation on a Bethe lattice with coordination number z.[21] We can present a simple intuitive argument to show that this percolation problem is nearly identical to LRBP in the mean-field regime. Consider the construction of a single cluster in LRBP. Figure 2 shows an example of the first two steps of this process on a two-dimensional lattice (although we note that this argument holds for arbitrary d). Initially, all bonds are closed. First we open a bond between two sites labeled A and B (Fig. 2a). We then open bonds between A and its remaining $z-1$ neighbors with probability $p \lesssim 1/(z-1)$. As a result, most bonds will remain closed, since the probability of k open bonds $P(k)$ is given by $P(k) = e^{-1}/k!$ (see Appendix). Suppose we find that the two bonds connecting A to sites C and D are open (Fig. 2b). Next we proceed to open bonds between B and its $z-1$ neighbors. Since in this example C happens to be a common neighbor of both A and B, there is a chance of a three-bond loop forming between the sites A, B, and C. However, since $P(k)$ is independent of z and $p \sim 1/(z-1)$, we can make the probability of an open bond connecting B and C as small as we wish by increasing the value of z. In this sense we can regard the neighbors of A as being distinct from those

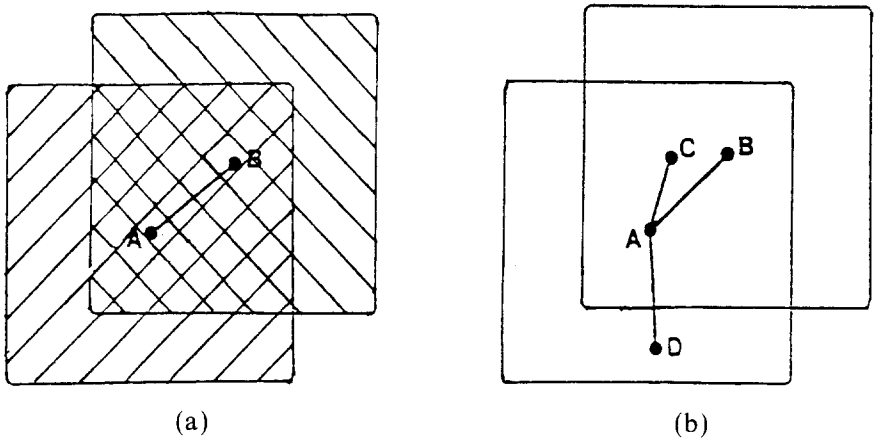


Fig. 2. (a) Construction of a cluster begins with the opening of a bond between sites A and B. The boxes indicate the neighborhoods of these two sites. R is large, so that the lattice spacing is too fine to be resolved in the diagram. (b) Bonds are opened between A and its remaining $z-1$ neighbors with probability $p \lesssim 1/(z-1)$. As a result, two other sites B and C are connected to A.

of B. We can continue to construct the cluster by repeating the "opening of bonds" process for every new site that we add. At some stage it is quite likely that there will be several sites lying within the neighborhood of the current growth site which already belong to the cluster. As long as their number is much smaller than z, our "distinct neighbors" approximation will still be valid. We could have duplicated this cluster construction method on a Bethe lattice without ever having to worry about all this; the $z-1$ offspring of every site are always distinct from those of other sites. It appears then that as long as the "distinct neighbor" approximation in LRBP holds, the intrinsic structure of clusters in both percolation problems will be identical.

It is important to note that the Bethe lattice we consider will have, in each generation, $z-1$ possible bonds of different lengths out of each vertex. In Euclidean space these bonds can cross (i.e., pass through each other). Such crossings do not, of course, generate loops and clearly will have no effect on the critical exponents, including the fractal dimension. They will, however, have a profound effect on the Hausdorff dimension.

The preceding arguments are substantiated by the simulation data in Figs. 3–5. We have grown large clusters in the LRBP regime using a

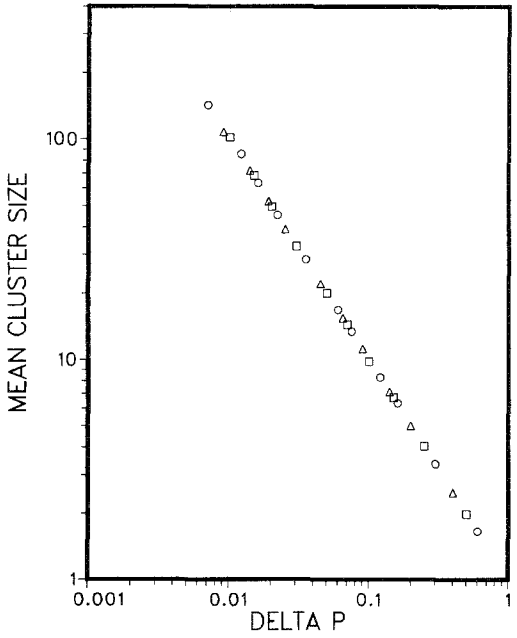


Fig. 3. Mean cluster size versus Δp for dimensions (○) two, (△) three, and (□) five when $R = 1000$. Each data point is the average size of 10,000 clusters. Error bar value: ± 0.02. Slope of graph: -1.001 ± 0.003.

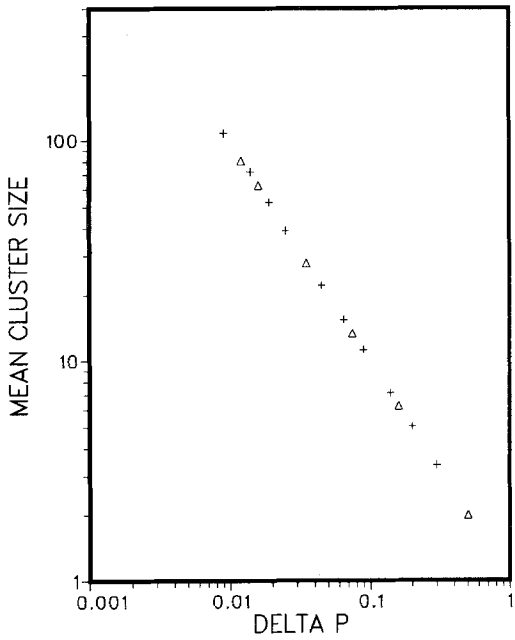


Fig. 4. Mean cluster size versus Δp for (+) $R = 1000$ and (△) $R = 10{,}000$. $d = 3$ in this plot; however, the result is independent of d. Each data point is the average size of 10,000 clusters. Error bar value: ± 0.03. Slope of graph: -0.995 ± 0.006.

modified Leath algorithm (see Appendix). In Fig. 3 we show the mean cluster size versus Δp in a log–log plot for $d = 2$, 3, and 5. Figure 4 is the same plot for two values of R. The plots are identical. This indicates that the mean cluster size is independent of R and d in both the amplitude and the scaling power as long as we remain in the mean-field regime. The amplitude appears to be unity, as does the value of γ. Denoting the mean cluster size as $S(p)$, we have

$$S(p) = (\Delta p)^{-1} \tag{2.1}$$

This is exactly the same form as the mean cluster size for bond percolation on a Bethe lattice in the limit of large coordination number.[21] Figure 5 shows the data which determine the cluster distribution exponent τ. We have generated 10^6 clusters and then graphed the number composed of s sites versus s on a log–log plot. The ordinate values are proportional to the probability of a site belonging to a cluster of size s. This means that in terms of the number of s-site clusters per site (n_s), the number of clusters composed of s sites will scale as

$$sn_s \sim s^{-\tau+1} \tag{2.2}$$

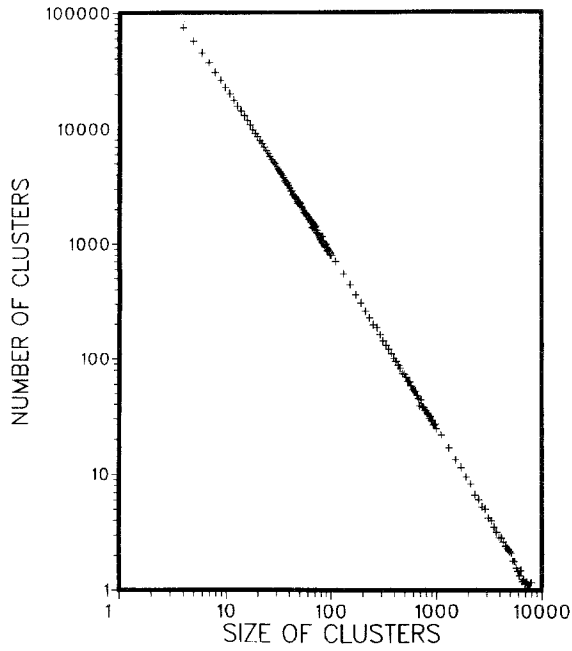


Fig. 5. Number of clusters versus size of clusters. $\Delta p = 0.007$, $R = 1000$, $d = 5$. A total of 10^6 clusters were generated to obtain this plot. Error bar value: ± 0.04. Slope of graph: -1.51 ± 0.02.

i.e., the slope of this plot should be $1 - \tau$. Quite accurately we find the value of the slope to be $-3/2$, indicating that τ has the mean-field value of 5/2. Again the result is independent of both R and d as long as we remain in the mean-field regime.

In other percolation problems, knowledge of γ and τ is enough information to determine the others by means of the scaling relations. However, since some of these involve d (i.e., the hyperscaling relations), this is not permitted in the mean-field regime of LRBP. Mean-field critical exponents only obey the hyperscaling relations when $d = 6$. Consequently, our measurements of γ and τ allow us to calculate α, β, and δ but not ν or η. In order to determine the remaining exponents, we must measure one of these last two. Our choice is the connectedness length exponent ν. We have taken the average value of the radius of gyration of several large clusters and graphed them as a function of Δp on a log–log plot. Figure 6 shows the data for the dimensions 2, 3, and 5. It is clear that ν has the mean-field value 1/2 and is independent of d.

We can also argue that ν should equal 1/2 from purely geometrical considerations. Consider the problem of determining the scaling of the con-

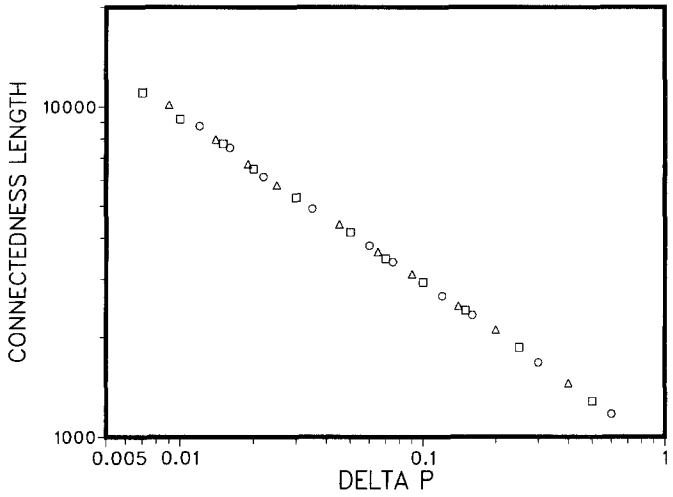


Fig. 6. Average radius of gyration of large clusters versus Δp for $d = (\bigcirc)\,2$, $(\triangle)\,3$, and $(\square)\,5$; $R = 1000$. Each data point is the average radius of gyration of ~ 40 clusters. Error bar value: ± 0.07. Slope of graph: 0.505 ± 0.019.

nectedness length ξ on a Bethe lattice below p_c. It is well known that one must imbed the lattice in an infinite-dimensional space to find the correct value of $\nu = 1/2$.[21] If instead we define ξ in terms of the number of generations spanned by the cluster or the "chemical distance,"[22] we find the exponent $\nu_{cd} = 1$. Now we also know that the number of generations in the backbone of a cluster the size of the connectedness length must also scale as $(\Delta p)^{-\nu_{cd}}$. But these clusters are identical to those in the mean-field regime of LRBP. Since the cluster has no loops and the large range R makes excluded-volume effects negligible, the backbone must describe a random walk. The modified Leath algorithm ensures that all backbone configurations are equally probable, which is the correct weighting for random walks (see Appendix). In any dimension the radius of gyration of a random walk scales as $n^{1/2}$, where n is the number of steps.[21] Therefore

$$\xi \sim n^{1/2} \sim [(\Delta p)^{-\nu_{cd}}]^{1/2} = \Delta p^{-1/2} \tag{2.3}$$

so that ν equals 1/2 in the classical regime of LRBP.

The connectedness length must also scale with the range of the bonds. Since all of the quantities we have measured have been independent of R, we expect it to enter in only as a prefactor of ξ. The role of R seems to be solely to provide enough neighboring sites in order for the probability of loops to be low enough so that the clusters can exist as branched objects in the mean-field regime. Figure 7 shows a graph of the average radius of

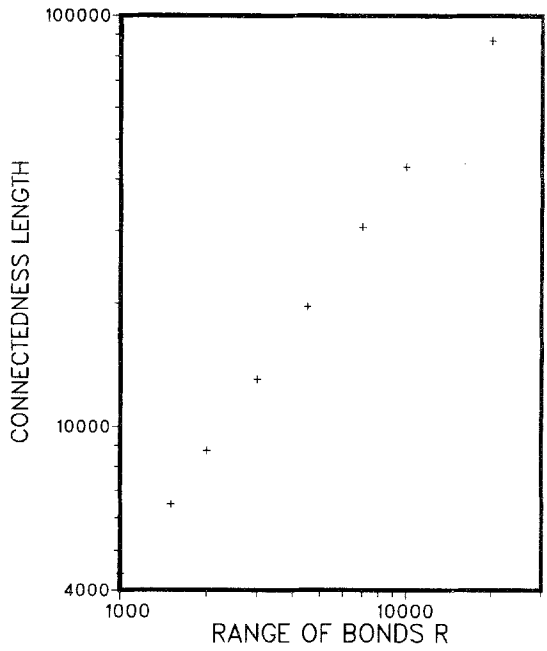


Fig. 7. Connectedness length as a function of R for $\Delta p = 0.05$, $d = 3$. Each data point is averaged from ~ 40 clusters. Error bar value: ± 0.10. Slope of graph: -0.997 ± 0.048.

gyration of large clusters at a fixed value of Δp versus the range of the bonds. As expected, we find that the connectedness length scales simply as R. We arrive at the scaling form of ξ:

$$\xi \sim R(\Delta p)^{-1/2} \tag{2.4}$$

The measurements of γ, τ, and ν provide strong evidence that there is a mean-field regime in LRBP. We know, however, that this regime cannot continue all the way to p_c; there will be a crossover to true asymptotic scaling below six dimensions. The Ginzburg criterion is commonly used to estimate where crossover occurs. This is a ratio of the mean square magnitude of fluctuations in a region the size of the connectedness length (i.e., ξ^d) to the square of the magnitude of the order parameter. As long as the ratio is small, the system will exhibit mean-field behavior. Below p_c the quantity that scales like the order parameter is P_{s*}, the probability that a site belongs to a cluster spanning a region of length ξ.[23] It follows that the number of sites within a region ξ^d which belong to large clusters must scale as $P_{s*}\xi^d$. The mean square fluctuations in this quantity are proportional to the second moment of the distribution, which is the mean cluster size $S(p)$

multiplied by ξ^d. The Ginzburg criterion[24] asserts that as long as the inequality

$$\xi^d S(p)/(\xi^d P_{s*})^2 \ll 1 \tag{2.5}$$

is satisfied, the system will exhibit classical scaling. Substituting the appropriate scaling quantities into (2.5), we have

$$R^d(\Delta p)^{\gamma+2\beta-d\nu} \gg 1 \tag{2.6}$$

or, since in the mean-field regime we have $\gamma = 1$, $\beta = 1$, $\nu = 1/2$,

$$R^d(\Delta p)^{3-d/2} \gg 1 \tag{2.7}$$

We will need this expression in the next section. For now we point out that the Ginzburg criterion is always well satisfied in our LRBP simulations. In our worst case $d=2$, $R=1000$, $\Delta p = 0.01$, so that the lhs of Eq. (2.7) is equal to 100.

3. THE BREAKDOWN OF HYPERSCALING AND THE CROSSOVER TO THE ASYMPTOTIC REGIME

In the preceding section we found mean-field values for the exponents γ, τ, and ν. It follows from the scaling relations that all of the critical exponents in this regime are classical. This result is independent of hyperscaling; we can use scaling relations that do not involve the dimension of space to obtain the rest of the exponents. In any dimension other than six the hyperscaling relations are therefore violated. We suggest that this breakdown of hyperscaling is intimately related to the crossover from classical to asymptotic scaling in $d<6$ dimensions.

We first consider a measurement which will lead us to the geometrical behavior underlying the breakdown of hyperscaling. Suppose we attempt to determine the exponent β from our simulations. We know that β must be equal to the mean-field value 1. In Section 2 we stated that the quantity which scales as $(\Delta p)^\beta$ (hence the singular part of the first moment of the cluster distribution) when $p < p_c$ is the fraction of sites within a volume ξ^d belonging to large clusters. Naively, one might assume that this quantity is proportional to the density of a single large cluster. In order to check the validity of this assumption, we plot the log of the average density of mass within clusters whose radius of gyration is larger than $R(\Delta p)^{-1/2}$ versus the log of Δp (Fig. 8). We find accurate values for the slope which do not correspond to the expected value of β and are dimensionally dependent. In two, three, and five dimensions they are -1, $-1/2$, and $+1/2$, respectively;

the slope increases by 1/2 each time we increment d. We conclude that the usual interpretation of β is incorrect in the mean-field regime of LRBP.

This anomalous behavior may be resolved by noting that we have tacitly assumed that the number of large clusters within a region ξ^d does not vary with Δp. In standard nearest-neighbor percolation this is indeed

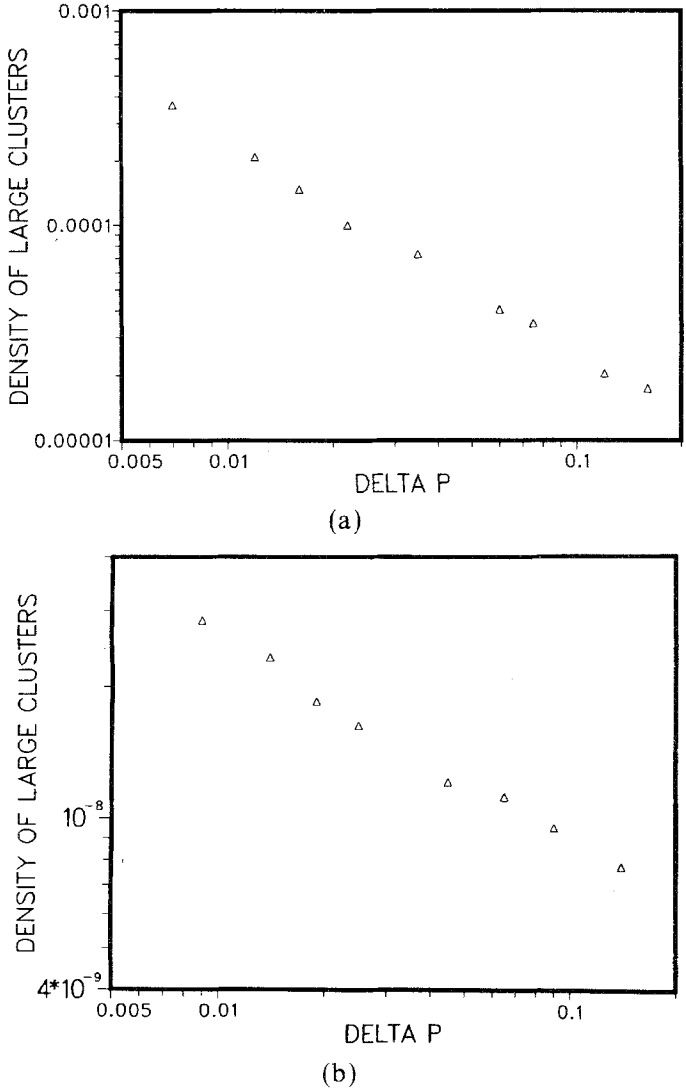
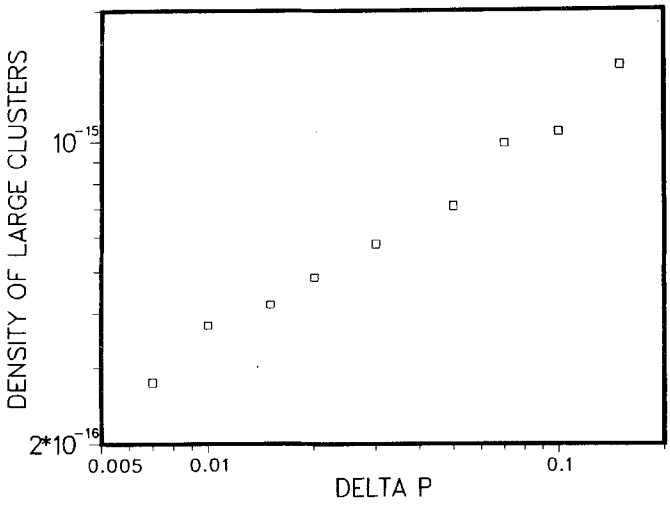


Fig. 8. Density of a single cluster versus Δp. Each data point is the average density of ~ 40 clusters. Error bar value: ± 0.15. (a) $d=2$, slope -0.98 ± 0.05. (b) $d=3$, slope -0.47 ± 0.06. (c) $d=5$, slope 0.51 ± 0.05. $R=1000$ for all plots.



(c)

Fig. 8 (*continued*)

the case; the fraction of sites belonging to large clusters is therefore proportional to the density of a single large cluster. However, if the number of these clusters were to scale as some nonzero power of Δp, it would be necessary to include it as a factor of the density of a single cluster in order to determine the correct scaling of the overall fraction of sites belonging to large clusters.[3] Let us denote the density of a single cluster by ρ_1 and the overall fraction of sites belonging to large clusters as ρ_T. Since the number of large clusters N_c in a volume ξ^d scales as

$$N_c \sim (\Delta p)^{2-\alpha} \xi^d \tag{3.1}$$

we would then have

$$\rho_T \sim (\Delta p)^{\beta} \sim [(\Delta p)^{2-\alpha} \xi^d](\rho_1) \tag{3.2}$$

We claim that this is precisely what is happening in LRBP. Rearranging expression (3.2), we find that the density of a single cluster scales as

$$\rho_1 \sim \frac{(\Delta p)^{\beta}}{(\Delta p)^{2-\alpha} \xi^d} \tag{3.3}$$

After substituting the mean-field values of the exponents into Eq. (3.3) and using Eq. (2.4), we obtain

$$\rho_1 \sim R^{-d} (\Delta p)^{-2+d/2} \tag{3.4}$$

This relation correctly gives the exponents measured in Fig. 8.

The assertion that the number of large clusters in LRBP does not remain constant when we vary Δp is a physical way of saying that the hyperscaling relation

$$2 - \alpha - d\nu = 0 \tag{3.5}$$

is violated in dimensions other than the upper critical dimension ($d = 6$). Below six dimensions the number of large clusters scales as a positive power of Δp, so that it goes to zero at p_c. This suggests the following geometrical mechanism enabling the system to crossover from classical to true asymptotic scaling below $d = 6$. In standard percolation the connectedness length grows as we decrease Δp just enough to maintain a fixed average number of large clusters. All critical behavior must therefore inherently depend upon the dimension of space, a fact that is manifested in the satisfaction of the hyperscaling relations. In the mean-field regime of LRBP, however, ξ is independent of d. As we decrease Δp below six dimensions, ξ does not grow rapidly enough to keep the number of large clusters constant and so this quantity will decrease. Thus, the system maintains its mean-field properties by "sacrificing" a portion of the large clusters; as long as there are enough sites within a region ξ^d to allow for a classical distribution of clusters where there is more than a single large cluster, the system will remain in the mean-field regime. Mathematically, we require [using (3.1)]

$$(\Delta p)^{2-\alpha}\, \xi^d = R^d (\Delta p)^{2-\alpha-d\nu} \gg 1 \tag{3.6}$$

Or, using the scaling relation $2 - \alpha = \gamma + 2\beta$,

$$R^d (\Delta p)^{\gamma + 2\beta - d\nu} \gg 1 \tag{3.7}$$

This inequality is the Ginzburg criterion discussed in Section 2. Our interpretation is therefore consistent with standard crossover scaling analysis. When R is fixed, the system must cross over to true asymptotic scaling in d dimensions when $(\Delta p)^{3-d/2} \sim R^{-d}$. We note that according to (3.7), LRBP will behave classically for smaller Δp if we increase the value of R. Geometrically, the effect of increasing the value of R is to provide the system with more sites inside all regions the size of ξ. This allows for the formation of a greater number of large clusters at a given value of Δp. Hence, the larger we make the value of R, the more we can decrease Δp before there is an average of only one large cluster per region of size ξ^d.

4. FRACTAL PROPERTIES OF MEAN-FIELD LRBP CLUSTERS

We have defined d_f to be the exponent which determines the scaling between the mass of large clusters M_{LC} and the connectedness length:

$$M_{LC} \sim \xi^{d_f} \tag{4.1}$$

Since the density of a large cluster ρ_1 is $\xi^{d_f - d}$, using Eq. (3.3) we find

$$\xi^{d_f} \sim \frac{(\Delta p)^{\beta}}{(\Delta p)^{2-\alpha}\,\xi^{d}}\,\xi^{d} \tag{4.2}$$

Substituting the mean-field values of the exponents into (4.2) and employing Eq. (2.4) yields $d_f = 4$, a result independent of the dimension of space. We check this by plotting the mass of clusters whose radius of gyration is on the order of $R(\Delta p)^{-1/2}$ against ξ. Figure 9 shows such a plot for $d = 2$, 3, and 5. The resulting slope is the expected value to a high degree of accuracy.

Next we consider the scaling behavior of the smaller clusters. Specifically, we wish to determine how the mass of a cluster scales with its

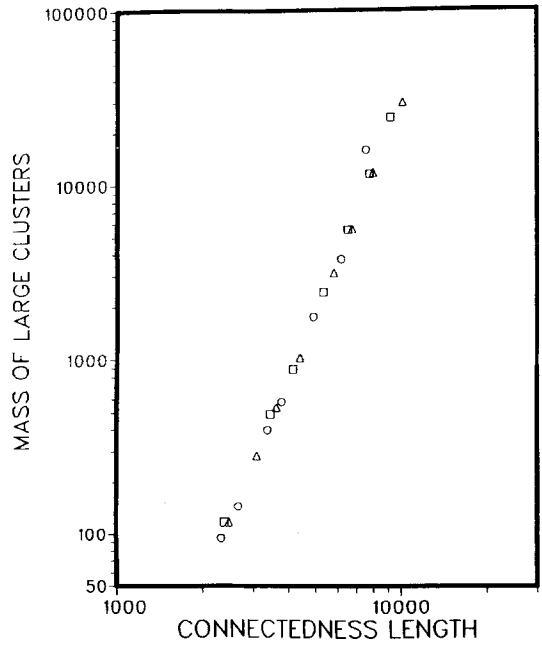


Fig. 9. Mass of large clusters versus connectedness lenth for $d = (\bigcirc)\,2$, $(\triangle)\,3$, and $(\square)\,5$; $R = 1000$. Each data point is the average mass of ~ 40 clusters. Error bar value: ± 0.05. Slope of graph: 3.96 ± 0.10.

radius of gyration R_G. If a small cluster is statistically the same as a large cluster generated at a bigger value of Δp, we will necessarily have the mass of any cluster scaling as R_G^4. Figure 10 shows a log–log plot of mass versus R_G for clusters generated at a fixed value of Δp. The data indicates that mass scales as R_G^4 for all clusters.

Finally, we investigate the Hausdorff dimension of LRBP clusters. Denoting the mass of a region inside a cluster by m and the length of that region by l, this means that

$$m(l) \sim l^{d_H} \tag{4.3}$$

The Hausdorff and fractal dimensions are equivalent in nearest-neighbor bond percolation because of the following special property: a small cluster is identical to a region of the same geometrical size within a larger cluster. Consequently, a region of length l inside a cluster will be the same as another cluster generated at a value of Δp where $\xi = l$. Therefore, the exponent d_f is necessarily the same as d_H. In four dimensions we can see that LRBP shares this property. Equation (3.4) shows that the density of large clusters is independent of Δp. The density of a region with length l is trivially the same as that of a large cluster when $\xi = l$; hence $d_f = d_H$.

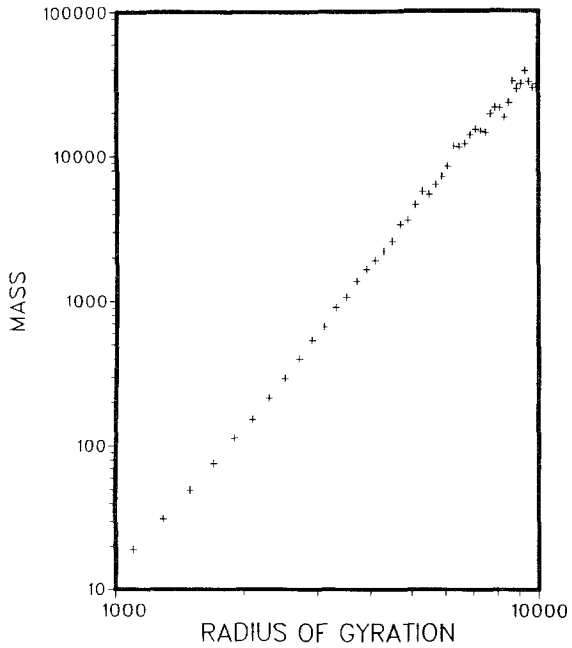


Fig. 10. Mass of clusters versus their radius of gyration for $d = 5$. Similar plots are obtained when $d = 2$ or 3. $R = 1000$. A total of 600,000 clusters were generated to make this plot.

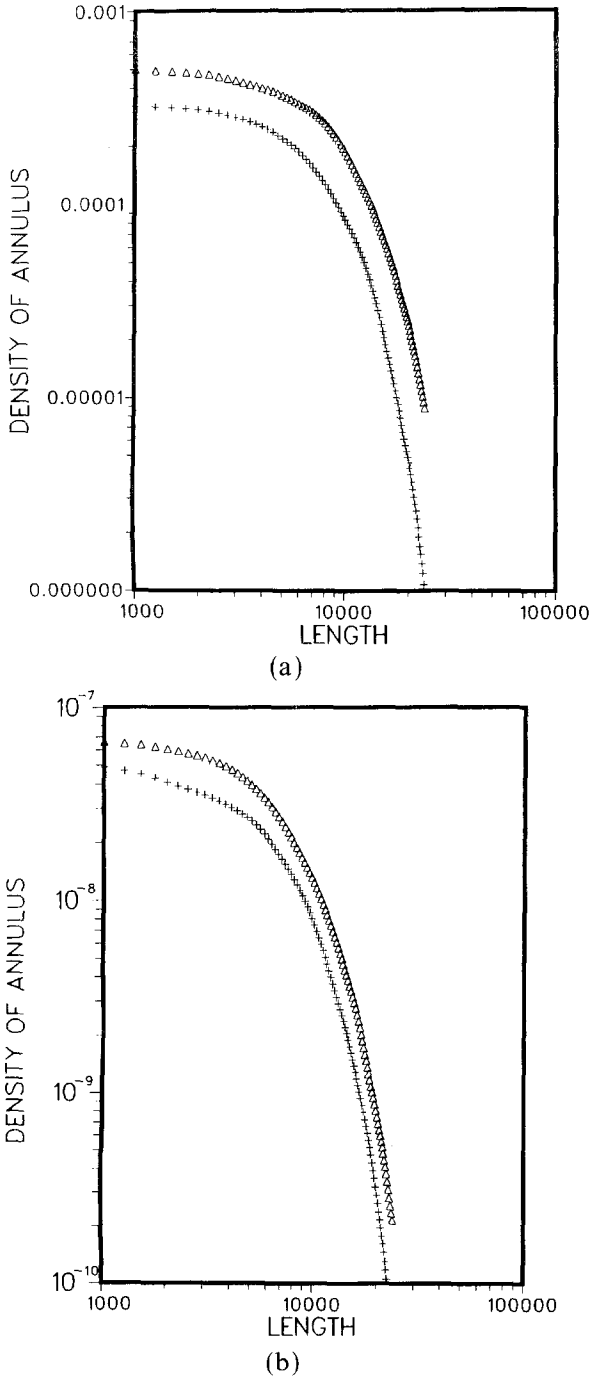
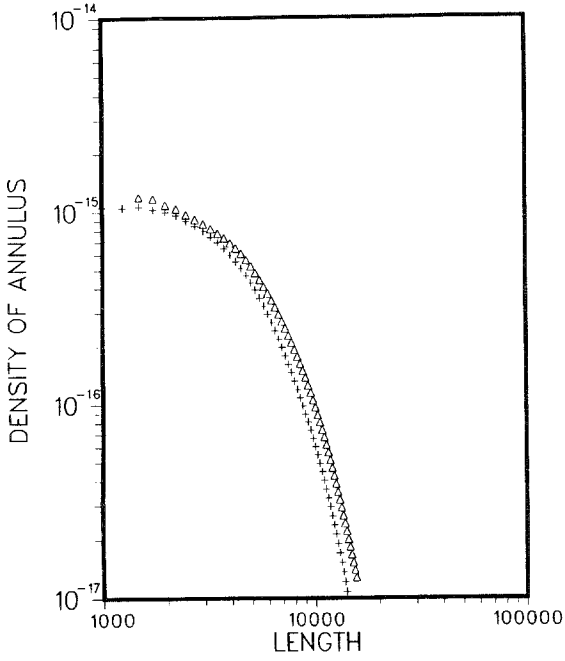


Fig. 11. Annulus plots of large clusters for different dimensions with $R = 1000$. (a) $d = 2$, (b) $d = 3$, (c) $d = 5$. Each annulus has a width of five lattice spacings. ($\triangle$) $\Delta p = 0.007$; ($+$) $\Delta p = 0.010$.



(c)

Fig. 11 (*continued*)

In dimensions other than four it is not obvious that $d_f = d_H$ in LRBP. Consider the process of growing a single cluster using the Leath algorithm. In short-range percolation most of the new growth sites are on the edges of the cluster, so that the central region remains unchanged. In LRBP the longer bonds can reach back from a peripheral site into the cluster's interior and seed it with new growth sites. This would cause the density to increase. For large enough d this type of "filling-in" effect could disappear due to the low probability of its occurrence.

We have made log–log plots of the density of annuli in large clusters versus the length of the annuli in an attempt to clarify these ideas. Recall that such a plot yields a slope of $d_H - d$. Due to computational limitations, we can not grow clusters large enough to accurately determine d_H; the finite-size effects are too large. However, there are trends in the data which give useful information. First we examine the behavior of LRBP clusters below $d = 4$. Figures 11a and 11b are annulus plots for large clusters in two and three dimensions, respectively. There is no indication of the slopes tending toward positive values as they would have to if $d_f = d_H$ (since $d_f = 4$). Instead, we see a trend to a slope of zero in the interior of the

clusters and a dramatic increase in the density of annuli as Δp is decreased. This suggests that they fill in uniformly as postulated above. For clusters in two and three it appears that $d_f \neq d_H$; instead, d_H is simply d.

Next we look at LRBP clusters in dimensions $d > 4$. One possibility is that $d_f = d_H$ so that $d = 4$ serves as a crossover dimension in the fractal behavior of the clusters. This point of view is supported by certain features of the annulus plot in five dimensions (Fig. 11c). The finite-size effects are very pronounced; most of the plot is simply a falloff effect due to the edge of the clusters. However, at small lengths the curves superimpose upon one another, indicating that the interior structure of large clusters does not depend upon Δp. The behavior of this plot is markedly different than Figs. 11a and 11b, where the interior structure changes radically. Recall also the data in Fig. 8c for the overall density of large clusters in five dimensions. As Δp decreased we saw that the density also decreased. If it is true that these clusters have an invariant inner structure, then it is impossible for their density to uniformly decrease as it would have to if $d_H = d$. In fact, this invariance would mean that a region of length l inside a large cluster is the same as another cluster generated at a value of Δp where $\xi = l$. This is precisely the geometrical property in short-range percolation which allows us to conclude that $d_f = d_H$. Although the above arguments seem plausible, our data are not good enough to test them thoroughly. Also, field theory arguments seem to indicate that $d_f = d$ in all dimensions.[25] More work must be done to clarify this problem.

In Section 3 we argued that the violation of hyperscaling is due to the nontrivial dependence of the number of large clusters on Δp. Presumably, then, if we could artificially fix the number of large clusters, hyperscaling would be restored. This is possible because we can make R an additional parameter. As a result, we have a peculiar situation where hyperscaling holds (but only because that is the condition we demand) and both ν and d_f depend upon the dimension of space. Setting the number of clusters [Eq. (3.1)] equal to a constant gives a scaling relation between R and Δp:

$$R \sim (\Delta p)^{1/2 - 3/d} \tag{4.4}$$

The connectedness length becomes

$$\xi = R(\Delta p)^{-1/2} = (\Delta p)^{-3/d} \tag{4.5}$$

so that

$$\nu = 3/d \tag{4.6}$$

Substituting into (4.2), we find

$$d_f = 2d/3 \tag{4.7}$$

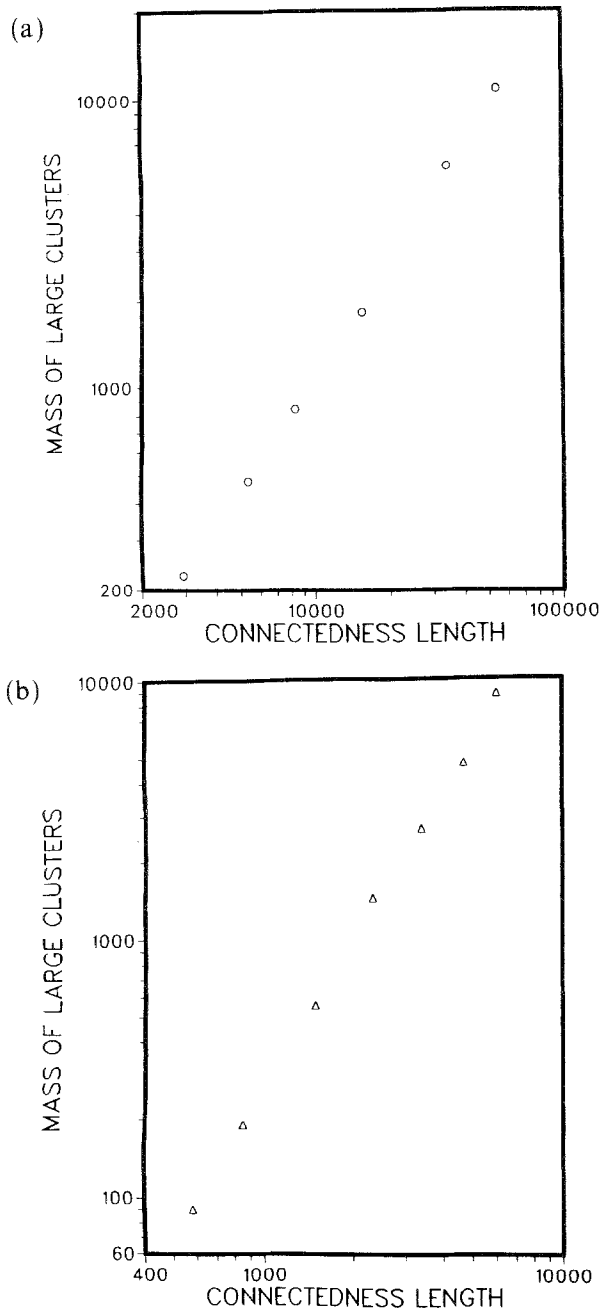


Fig. 12. Mass of large clusters versus connectedness length for (a) $d=2$ (slope 1.33 ± 0.04) and (b) $d=3$ (slope 1.96 ± 0.05), where $R^d(\Delta p)^{3-d/2}$ is held constant. Error bar value: ± 0.10.

Note that the same result may be obtained using Kadanoff scaling arguments[24] if we require mean-field values for γ and β independent of d. Denoting the ghost field as h and the length of a block by L, we have for the rescaled fields

$$\Delta p' = L^y \Delta p \tag{4.8}$$

$$h' = L^x h \tag{4.9}$$

Using standard techniques, we obtain the scaling relations for β and γ in terms of x, y, and d:

$$\beta = (d-x)/y \tag{4.10}$$

$$\gamma = (2x-d)/y \tag{4.11}$$

Requiring β and γ to be their mean-field values, we find

$$x = 2d/3 \tag{4.12}$$

$$y = d/3 \tag{4.13}$$

With the assumption of hyperscaling, we have $x = d_f$ and $y^{-1} = \nu$. These results agree with Eqs. (4.6) and (4.7).

Figure 12 shows log–log plots in both $d=2$ and $d=3$ of the mass of large clusters versus the connectedness length, where we have kept the number of large clusters fixed according to the above prescription. As expected from the analysis above, we find slopes of 4/3 and 2, respectively.

5. CONCLUSION

We have shown that, in contrast to long-range site percolation,[6,7] LRBP does have a mean-field regime. This regime is characterized by

$$N_c \equiv R^d \Delta p^{3-d/2} \gg 1 \tag{5.1}$$

which is obtained from the Ginsburg criterion. Here N_c is the number of clusters the size of the connectedness length ξ.

We have also shown that in LRBP, $(\Delta p)^\beta$ is not the density of one infinite cluster but is, in fact, the density of sites in all N_c infinite clusters. Consequently, the density of sites in one infinite cluster is given by

$$(\Delta p)^\beta / N_c \tag{5.2}$$

The breakdown of hyperscaling in LRBP is associated with the fact that N_c scales with Δp. This is consistent with the results of Coniglio[3] and Aharony *et al.*,[4] who studied nearest-neighbor bond percolation for $d>6$, where the same behavior was encountered. However, we can use the bond range R to restore hyperscaling. This is done by scaling R in such a way

that N_c remains fixed when Δp is varied. In this way hyperscaling is restored and the fractal dimension is given by

$$d_f = d - \beta/\nu = 2d/3 \tag{5.3}$$

These results can also be obtained from a renormalization group (RG), i.e., Kadanoff rescaling, approach. This suggests that the bond or R rescaling defined above, when used in an RG context, generates a different fixed point than an RG using fixed R. This behavior was also observed by Green in his investigation of thermal models with long-range interactions.[26]

Finally, the exponent d_f of our clusters seems to be well understood; however, the Hausdorff dimension d_H is not. Field theory calculations[17] appear to show that $d_H = d$ for all d. However, our data, as explained above, indicate that $d_H = d$ only for $d \leqslant 4$. This will require further investigation.

APPENDIX. THE MODIFIED LEATH ALGORITHM

A single cluster is generated by the following method.[26] We select two sites to initially belong to the cluster. One of them is the origin and the other is chosen at random within range of the origin. We say that each of these sites is now a "growth site." By a Monte Carlo process we will now attempt to add new sites to the cluster by opening bonds with probability p between each of these sites and the remaining $z-1$ neighbors. Since z is typically equal to 10^6, it is too inefficient for us to generate a random number to determine the status of each bond. Instead, we use a trick due to Gawlinski and Stanley[28]; we find the number of open bonds by randomly sampling a Poisson distribution. The mean number of open bonds λ will be $p(z-1)$ or, since $p = p_c(1-\Delta p)$ and $p_c = 1/(z-1)$, we have $\lambda = 1 - \Delta p$. The probability of k open bonds is then $\lambda^k e^{-\lambda}/k!$. Having determined k, we randomly select that number of sites within the interaction range of the growth site to be new members of the cluster. The growth site is now termed a "dead site." We repeat this process for new sites as they are added until the cluster stops growing; i.e., all sites are dead.

There are two things to note about this algorithm. First, it is apparent that all of our clusters have twice as many sites as those in the true distribution. This will not affect scaling, but it will change the amplitude of various quantities (e.g., the mean cluster size). Also, it is possible to select a new site in the process of adding sites which happens to be a dead site, thereby invalidating the whole algorithm. However, for large R this occurs only very rarely. This is the "distinct neighbor" approximation discussed in Section 2. When we have finished constructing a cluster we can go back

and check to make sure that no loops have formed. Our algorithm is therefore tailored for the mean-field regime and will not work in the asymptotic regime.

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